

Linking diagrams for free

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Linking diagrams with path composition are ubiquitous, for example: Temperley-Lieb and Brauer monoids, Kelly-Laplaza graphs for compact closed categories, and Girard's multiplicative proof nets. We construct the category $\text{Link} = \mathbf{Span}(\text{iRel})$, where iRel is the category of injective relations (reversed partial functions) and show that the aforementioned linkings, as well as Jones-Martin partition monoids, reside inside Link . Path composition, including collection of loops, is by pullback. Link contains the free compact closed category on a self-dual object (hence also the looped Brauer and Temperley-Lieb monoids), and generalises partition monoids with partiality (vertices in no partition) and empty- and infinite partitions. Thus we obtain conventional linking/partition diagrams and their composition “for free”, from iRel .

1 Introduction

Write Brau^b for the category of loopless Brauer linkings [Bra37]:

- *Objects* X, Y, \dots are finite sets, whose elements we call **vertices**.
- *Morphisms*. A **loopless Brauer linking** $X \rightarrow Y$ is an equivalence relation on the disjoint union $X + Y$ whose every class is a pair (2 vertices).
- *Composition* is path composition: the composite $SR : X \rightarrow Z$ of $R : X \rightarrow Y$ and $S : Y \rightarrow Z$ is the restriction to $X + Z$ of the transitive closure $(R + S)^*$ of $R + S \subseteq X + Y + Z$.¹ See Figure 1.

The **loopless Brauer monoid** Brau_n^b is the subcategory of Brau^b on $\{1, \dots, n\}$.²

Write Brau for the category of looped Brauer linkings, on the same objects:

- *Morphisms*. A **looped Brauer linking** $X \rightarrow Y$ is a pair $\langle k, R \rangle$, denoted $\delta^k R$, comprising a loopless Brauer linking $R : X \rightarrow Y$ a **loop count** $k \in \mathbb{N} = \{0, 1, \dots\}$.
- *Composition* is path composition, collecting loops: $(\delta^l S)(\delta^k R)$ is $\delta^{l+k+\lambda}(SR)$ where SR is the composite in Brau^b and λ is the number of **loops** formed during the construction of SR , that is, classes of $(R + S)^* \subseteq X + Y + Z$ which are entirely within Y . See Figure 2.

The **looped Brauer monoid** Brau_n is the subcategory of Brau on $\{1, \dots, n\}$.³ The category Brau is (equivalent to) the free compact closed category on a self-dual object [KL80, Abr05]. There is a

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¹To avoid clutter we assume here (without loss of generality, by renaming vertices) that canonical injections $Q_i \rightarrow Q_1 + Q_2$ are inclusions. In other words, we assume X, Y and Z are disjoint, and that every $+$ is a union \cup .

²I.e., the monoid Brau_n^b is the homset $\text{Brau}^b(\{1, \dots, n\}, \{1, \dots, n\})$, with composition as multiplication. Although [Bra37] considered only monoids, collecting them into a category is obvious and trivial.

³ Brau_n is the submonoid of the Brauer algebra over n [Bra37] generated (under multiplication in the algebra) by $\{\delta^0 R : R \in \text{Brau}_n^b\}$ and $\delta^1 i$, where i is the identity in Brau_n^b .

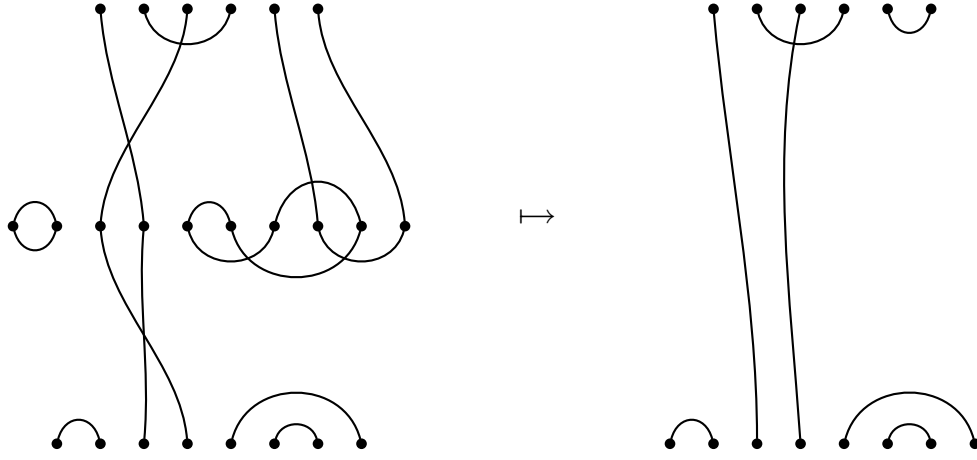


Figure 1: Example of composition in the category Brau^b of loopless Brauer linkings. Each equivalence class $\{x, y\}$ is depicted as a “link” on x and y .

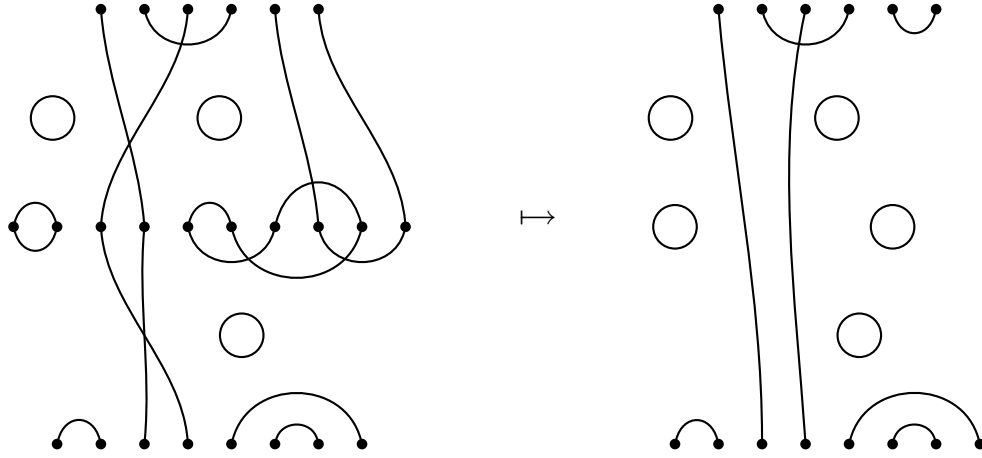


Figure 2: Example of composition in the category Brau of looped Brauer linkings. The two input linkings are $\delta^2 R$ (upper) and $\delta^1 S$ (lower), where R and S are the loopless linkings in Figure 1. The output linking $(\delta^1 S)(\delta^2 R)$ is $\delta^5(SR) = \delta^{1+2+2}(SR)$, where SR is the output loopless linking in Figure 1, a composition which forms two new loops.

forgetful functor to both \mathbf{Brau} and \mathbf{Brau}^b from the category \mathbf{MLL} of unit-free multiplicative proof nets [Gir87], extracting leaves (literal occurrences) and axiom links.⁴

The separate treatment of paths and loops is ad hoc. We shall unify paths and loops, handling them simultaneously, and in so doing, obtain infinite generalisations of linkings.

Acknowledgement. Thanks to Robin Houston for feedback last summer on the prospect of extending pullbacks from injective relations to coherence spaces [Gir87] for a “sliced” notion of linking, enriched in commutative monoids. This is work in progress.

Many thanks to Vaughan Pratt for his ongoing support.

2 Generalised linkings: $\text{Link} = \text{Span}(\mathbf{iRel})$

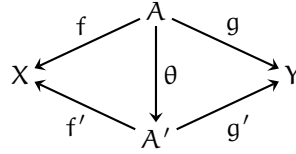
A binary relation $R : A \rightarrow Z$ (i.e., $R \subseteq A \times Z$) is **injective** if aRz and $a'Rz$ implies $a = a'$.⁵ Write \mathbf{iRel} for the category of sets and injective relations between them. Note that $\mathbf{iRel} = \mathbf{pFun}^{\text{op}}$, the opposite of the category of sets and partial functions.

A **linking** $X \rightarrow Y$ is a diagram

$$X \xleftarrow{f} A \xrightarrow{g} Y$$

in \mathbf{iRel} . Each $a \in A$ is a **link**, and the elements of X and Y are **vertices**. The vertex set $f(a) + g(a) \subseteq X + Y$ is the **footprint** of the link $a \in A$.⁶ If a vertex x is in the footprint of a , we simply say that x is in a , or a has/contains x . The injectivity requirement implies that no two links overlap (share a vertex). See Figure 3 for examples.

Just as graph theory treats graphs up to isomorphism, we identify linkings up to isomorphism, i.e., renaming of links. Formally, we identify linkings $X \xleftarrow{f} A \xrightarrow{g} Y$ and $X \xleftarrow{f'} A' \xrightarrow{g'} Y$ iff there exists a bijection $\theta : A \rightarrow A'$ such that $f'\theta = f$ and $g'\theta = g$.



⁴An object of \mathbf{MLL} is a unit-free multiplicative formula, a morphism $A \rightarrow B$ is a cut-free proof net on $A \multimap B$, and composition is by cut elimination. See e.g. [HG03, HG05]. The well-definedness to \mathbf{Brau}^b is trivial; the functor to \mathbf{Brau} is more subtle, being well-defined because proof net correctness ensures no loops arise during composition (i.e., $\lambda = 0$ in the definition of composition in \mathbf{Brau}).

⁵ aRz abbreviates $\langle a, z \rangle \in R$.

⁶For any binary relation $R : A \rightarrow Z$, the image $R(a)$ is $\{z \in Z : aRz \text{ for some } a \in A\} \subseteq Z$.

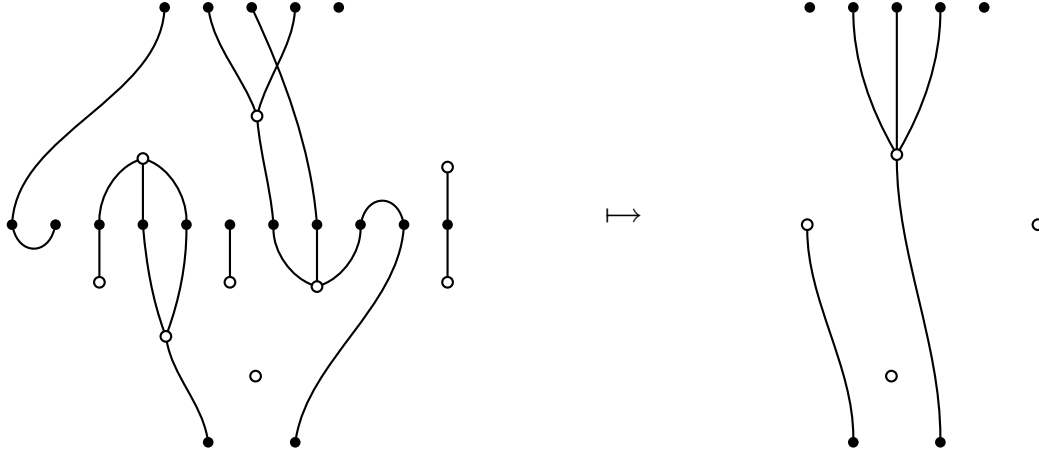


Figure 3: Examples of linkings and pullback-composition in $\text{Link} = \mathbf{Span}(\mathbf{iRel})$. A link is shown as a small circle, with its vertices attached by edges. We leave the circle implicit when a link has two vertices.

2.1 Composition by pullback

The composite $X \rightarrow Z$ of linkings $X \xleftarrow{f} A \xrightarrow{g} Y$ and $Y \xleftarrow{h} B \xrightarrow{k} Z$ is by pullback in \mathbf{iRel} :⁷

$$\begin{array}{ccccc}
 & & P & & \\
 & p \swarrow & & \searrow q & \\
 & A & & B & \\
 f \swarrow & & & & \searrow k \\
 X & & g \searrow & h \swarrow & Z \\
 & & Y & &
 \end{array} \tag{1}$$

Explicitly, the composite linking

$$X \xleftarrow{fp} P \xrightarrow{kq} Z$$

is defined as follows. To illustrate the definition as we proceed, we refer to the Brauer composition in Figure 2. There $X/Y/Z$ are the upper/mid/lower rows, and A/B are the upper/lower link sets.

A **synchronisation** $\langle \alpha, \beta \rangle$ is a pair of sets of links $\alpha \subseteq A$ and $\beta \subseteq B$ with the same footprint in the interface Y :

$$g(\alpha) = h(\beta) \tag{2}$$

For example, in Figure 2, if α comprises the three caps \frown of A , and β the first three cups \smile of B , then $\langle \alpha, \beta \rangle$ is a synchronisation with $f(\alpha) = g(\alpha) = \{y_1, y_2, y_5, y_6, y_7, y_9\} \subseteq Y$, where the y_i are the vertices of Y from left to right. (Note that this remains a synchronisation upon adding any number of loops to α and β , since loops have empty footprint in Y .) Henceforth identify a synchronisation $\langle \alpha, \beta \rangle$ (and more generally any pair $\langle \alpha, \beta \rangle$ of subsets $\alpha \subseteq A$ and $\beta \subseteq B$) with the corresponding

⁷Equivalently, pushout in $\mathbf{pFun} = \mathbf{iRel}^{\text{op}}$. The use of spans/pullbacks in this paper, together with equivalence up to isomorphism, should be compared with the standard use of cospans/pushouts for tangles and cobordisms.

subset $\alpha + \beta \subseteq A + B$ (thus identifying along the bijection⁸ $\mathcal{P}(A) \times \mathcal{P}(B) \cong \mathcal{P}(A + B)$, where $\mathcal{P}(C)$ denotes the powerset (set of subsets) of C).

A (generalised) **path** is a minimal non-empty synchronisation, where minimality is with respect to inclusion. There are 12 paths in Figure 2: seven singletons (the two loops in A , the loop in B , the cup of A , and the three caps of B), three doubletons (the short circuit formed on $\{y_1, y_2\}$ and the verticals through y_3 and y_4), one triplet (through y_8 and y_{10}), and one quadruplet (the long circuit through y_5, y_6, y_7, y_9).

Define the set P of links of the composite $X \xleftarrow{fp} P \xrightarrow{kq} Z$ as the set of all paths, and define $p : P \rightarrow A$ and $q : P \rightarrow B$ as the projections

$$p\langle\alpha, \beta\rangle = \alpha \quad (3)$$

$$q\langle\alpha, \beta\rangle = \beta \quad (4)$$

In Figure 2, p (resp. q) projects each path to its constituent links in the upper half A (resp. lower half B). The composite $fp : P \rightarrow X$ projects a path γ to the vertices (if any) in X which are on γ , and similarly for $kq : P \rightarrow Y$. In particular, for each of the five loops L (both the three singletons from the original linkings, and the two formed of multiple links), we have $fp(L)$ and $kq(L)$ empty.

See Figure 3 for a more general, non-Brauer example. An example of an infinite composition is depicted in Figure 4, illustrating why naive infinite generalisations of Brauer linkings do not work: an infinite chain of binary (two-vertex) links produces a unary (single-vertex) link. A finite variant is in Figure 5.

THEOREM 1 *The construction above defines pullbacks in $iRel$.*

Proof. Section 5. □

Write \mathbf{Link} for the category of linkings with this composition. In other words, $\mathbf{Link} = \mathbf{Span}(iRel)$, the span construction [Bén67] applied to $iRel$, with bicategorical structure collapsed to a category by taking morphisms (1-cells) up to isomorphism. That \mathbf{Link} is a category (with identities and associative composition) follows from the general features of the **Span** construction, saving considerable labour.

2.2 Loopless variant \mathbf{Link}^b

A **loop** is a link without vertices. Define \mathbf{Link}^b as the variant of \mathbf{Link} comprising the loopless linkings, discarding any loops formed during pullback composition. (Composition is associative since loops do not interact during pullback.) Write $(-)^b : \mathbf{Link} \rightarrow \mathbf{Link}^b$ for the functor which deletes loops (identity on objects). Note that \mathbf{Link}^b is not a subcategory of \mathbf{Link} , since composition of loopless linkings can generate loops.

3 Subcategories of \mathbf{Link} and \mathbf{Link}^b

We consider various subcategories of \mathbf{Link} and \mathbf{Link}^b , as summarised in Figure 6 and detailed below.

The categories \mathbf{Brau} and \mathbf{Brau}^b were defined at the start of Section 1. The categories \mathbf{Part} and \mathbf{Part}^b are the looped and unlooped **Jones-Martin partition categories** [Jon94, Mar94]⁹, defined

⁸More suggestively, $2^A \times 2^B \cong 2^{A+B}$, writing 2^C for $\mathcal{P}(C)$.

⁹As with the Brauer category, we have merely collected the monoids into categories in the obvious way.

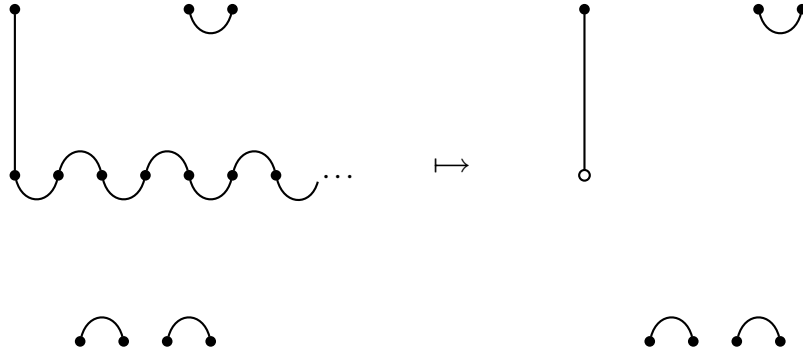


Figure 4: Example of the pullback-composition of infinite linkings in $\text{Link} = \mathbf{Span}(\text{iRel})$, with interface layer $Y = \{1, 2, \dots\}$. The entirety of Y is a synchronisation, and since it is minimal and non-empty, it is a path. Thus it shows up in the result of composition on the right, as a unary link (with a single vertex). This shows clearly why naive infinite generalisations of Brauer linkings do not work: an infinite chain of binary (two-vertex) links has produced a unary link. Figure 5 shows a corresponding example in which the interface layer is finite.

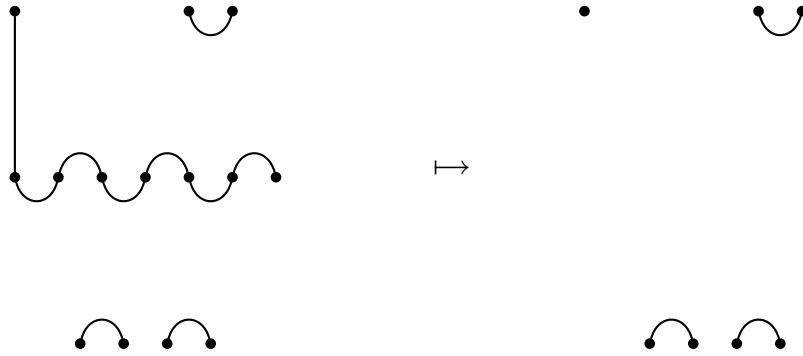


Figure 5: Analogous composition to Figure 4 in which the interface layer $Y = \{1, \dots, 7\}$ is finite. This time there is no non-empty synchronisation touching Y , so no link results from the interaction there. (Note that the lower-left input linking is not a Brauer linking, since it is partial: vertex 7 is in no link.)

$$\begin{array}{ccccccc}
(\mathbb{N}, +) & \subseteq & \text{TLieb} & \subseteq & \text{Brau} & \subseteq & \text{Part} & \subseteq & \text{Link} \\
& & \downarrow \text{\tiny b} & & \downarrow \text{\tiny b} & & \downarrow \text{\tiny b} & & \downarrow \text{\tiny b} \\
& & \text{TLieb}^b & \subseteq & \text{Brau}^b & \subseteq & \text{Part}^b & \subseteq & \text{Link}^b
\end{array}$$

		Object restriction	Morphism restriction				
			<i>loopless</i>	<i>finite</i>	<i>total</i>	<i>binary</i>	<i>planar</i>
	Link						
	Part	finite		✓	✓		
	Brau	finite		✓	✓	✓	
	TLieb	$\{1, \dots, n\}$		✓	✓	✓	✓
	$(\mathbb{N}, +)$	empty		✓	(✓)	(✓)	(✓)
	Link ^b		✓				
	Part ^b	finite	✓	(✓)	✓		
	Brau ^b	finite	✓	(✓)	✓	✓	
	TLieb ^b	$\{1, \dots, n\}$	✓	(✓)	✓	✓	✓

Figure 6: Various subcategories of Link and its loopless variant Link^b. *Total* means every vertex is in a link. *Binary* means every non-loop has exactly two vertices. The (✓) are implied ✓. Here $(\mathbb{N}, +)$ is the monoid of integers under addition, which is the subcategory of Link on the empty set. The functor $(-)^b$ deletes all loops. The categories Part^b, Brau^b and TLieb^b contain the standard (loopless) Jones-Martin partition-, Brauer- and Temperley-Lieb monoids, respectively.

exactly as Brau and Brau^b (verbatim), but dropping the restriction that every equivalence class is a pair. The conventional (loopless) partition monoid on n is the subcategory Part_n^b of Part^b on $\{1, \dots, n\}$.

The **Temperley-Lieb category** TLieb [TL71]¹⁰ is the subcategory of Brau on objects of the form $\{1, \dots, n\}$ for $n \geq 0$, and with only the planar¹¹ linkings (no crossings of links, *i.e.*, well-bracketed or “parenthetical” [Kau04, p. 63]). See [Abr07] for a concrete presentation. The category TLieb^b is the loopless variant of TLieb . The standard loopless Temperley-Lieb monoids are the subcategories of TLieb^b on the objects $\{1, \dots, n\}$.

Planar partition monoids can be defined by analogy with Temperley-Lieb monoids. For a nice exposition of each of the aforementioned monoids (and their algebras), with many diagrams and examples, see [HR05].

4 Geometry of interaction “for free”

Let MLL denote the category of multiplicative proof nets [Gir87], with unit-free formulas as objects, a morphism $X \rightarrow Y$ as a cut-free proof net on $X \multimap Y$, and composition by cut elimination. Thus a proof net is a linking on leaves (literal occurrences) which satisfies a correctness criterion, and composition is path composition.¹² The forgetful functor $L^b : \text{MLL} \rightarrow \text{Brau}^b$ extracts the leaves (forgetting the underlying parse tree structure of the formulas) and the links between them. Due to the correctness criterion on proof nets, loops never arise during composition, thus there is also a forgetful functor $L : \text{MLL} \rightarrow \text{Brau}$, and the following diagram commutes.

$$\begin{array}{ccccc} \text{MLL} & \xrightarrow{L} & \text{Brau} & \subseteq & \text{Link} \\ & \searrow L^b & \downarrow b & & \downarrow b \\ & & \text{Brau}^b & \subseteq & \text{Link}^b \end{array}$$

Having composed the linkings of proof nets $A \multimap B$ and $B \multimap C$ in Link by $i\text{Rel}$ pullback, we can draw the resulting linking on $A \multimap C$, to obtain the composite in MLL . Thus all computation happens inside Link , so we have geometry of interaction [Gir89] “for free”, via $i\text{Rel}$.

Work in progress aims to use pullbacks of coherence spaces [Gir87], an extension of $i\text{Rel}$, to obtain a multiplicative-additive geometry of interaction “for free”.

5 Proof of Theorem 1

A binary relation $R : A \rightarrow Z$ is **total** if the image $R(a) \subseteq Z$ is non-empty for all $a \in A$.

LEMMA 1 *An $i\text{Rel}$ morphism is monic¹³ iff it is total.¹⁴*

Proof. Suppose $m : A \rightarrow Z$ is total. Let $f, g : W \rightarrow A$ with $mf = mg$. If $f \neq g$ there exist $w \in W$ and $a \in A$ with $wf a$ but not $wg a$ (exchanging f and g , if necessary). Since m is total, there exists

¹⁰See footnote 9.

¹¹We assume vertices $1, \dots, n$ are ordered in the plane.

¹²See *e.g.* [HG03, HG05].

¹³Recall that a morphism $m : A \rightarrow Z$ is *monic* if $mf = mg$ implies $f = g$ for all objects W and $f, g : W \rightarrow A$ [Mac71].

¹⁴Dually, and perhaps more intuitively obvious, a partial function is *epic* (in pFun) iff it is surjective.

$z \in Z$ with $a m z$. Thus $w(mf)z$, so $w(mg)z$, hence there exists $a' \in A$ with $w g a' m z$. Since not $w g a$, we have $a' \neq a$, but then $a m z$ and $a' m z$ contradicting injectivity. Thus $f=g$, so m is monic.

Conversely, suppose $m : A \rightarrow Z$ is not total. Then there exists $a \in A$ such that $m(a) = \emptyset$. Let $W = \{w\}$, $f(w) = \emptyset$ and $g(w) = \{a\}$. Then $mf = mg$ (both empty) yet $f \neq g$, so m is not monic. \square

LEMMA 2 (STABILITY) *Injective relations preserve unions and intersections: for any $R : A \rightarrow Z$ in $iRel$ and subsets $\alpha_i \subseteq A$ for each i in some indexing set I ,*

$$R \left(\bigcup_{i \in I} \alpha_i \right) = \bigcup_{i \in I} R(\alpha_i) \quad (5)$$

$$R \left(\bigcap_{i \in I} \alpha_i \right) = \bigcap_{i \in I} R(\alpha_i) \quad (6)$$

Proof. (5). A trivial property of binary relations (injectivity not required).¹⁵

(6). Suppose $z \in R(\bigcap \alpha_i)$, i.e., aRz for some $a \in \bigcap \alpha_i$. Then $a \in \alpha_i$ for all i , hence $z \in R(\alpha_i)$ for all i , so $z \in \bigcap R(\alpha_i)$. Conversely, suppose $z \in \bigcap R(\alpha_i)$, i.e., $z \in R(\alpha_i)$ for all i . Then for each $i \in I$ there exists $a_i \in \alpha_i \subseteq A$ with $a_i R z$. By injectivity, $a_i = a_j = a$ for all $i, j \in I$, hence $a \in \bigcap \alpha_i$. Thus $z \in R(\bigcap \alpha_i)$, since aRz . \square

Write $\alpha \uplus \beta$ for $\alpha \cup \beta$ when $\alpha \cap \beta = \emptyset$, and more generally, write $\biguplus_{i \in I} \alpha_i$ for $\bigcup_{i \in I} \alpha_i$ when $\alpha_i \cap \alpha_j = \emptyset$ for all distinct $i, j \in I$.

COROLLARY 1 *Injective relations preserve disjoint unions: with R as in the previous lemma,*

$$R \left(\biguplus_{i \in I} \alpha_i \right) = \biguplus_{i \in I} R(\alpha_i) \quad (7)$$

Proof. Immediate from (5) and (6). \square

COROLLARY 2 *Injective relations preserve inclusion and subtraction: if $R : A \rightarrow Z$ in $iRel$ and $\alpha, \beta \subseteq A$ then¹⁶*

$$\alpha \subseteq \beta \implies R(\alpha) \subseteq R(\beta) \quad (8)$$

$$R(\alpha \setminus \beta) = R(\alpha) \setminus R(\beta) \quad (9)$$

Proof. (8) is trivial (for any binary relation), and (9) is immediate from the properties above:

$$R(\alpha) = R((\alpha \setminus \beta) \uplus (\alpha \cap \beta)) \stackrel{(6,7)}{=} R(\alpha \setminus \beta) \uplus (R(\alpha) \cap R(\beta)) \quad (10)$$

hence

$$R(\alpha \setminus \beta) = R(\alpha) \setminus (R(\alpha) \cap R(\beta)) = R(\alpha) \setminus R(\beta) \quad \square$$

¹⁵ $b \in R(\bigcup \alpha_i)$ iff $\exists a \in \bigcup \alpha_i. aRb$ iff $\exists i \in I, a \in \alpha_i. aRb$ iff $\exists i \in I, z \in R(\alpha_i)$ iff $z \in \bigcup R(\alpha_i)$.

¹⁶ $\alpha \setminus \beta = \{a \in \alpha : a \notin \beta\}$.

Refer once again to the diagram (1). Recall that we identify a pair $\langle \alpha, \beta \rangle$ of subsets $\alpha \subseteq A$ and $\beta \subseteq B$ with $\alpha + \beta \subseteq A + B$. Intersection, union and inclusion of synchronisations are defined via this identification. Write $h(\langle \alpha, \beta \rangle) = h(\alpha)$ and $g(\langle \alpha, \beta \rangle) = g(\beta)$. Thus $\sigma \subseteq A + B$ is a synchronisation iff

$$h(\sigma) = g(\sigma) \quad (11)$$

LEMMA 3 *Synchronisations are closed under union, intersection and subtraction:*

- (a) *if S is a set of synchronisations then $\bigcap S$ and $\bigcup S$ are synchronisations;*
- (b) *if σ and τ are synchronisations then $\sigma \setminus \tau$ is a synchronisation.*

Proof.

$$g\left(\bigcap S\right) \stackrel{(6)}{=} \bigcap_{\sigma \in S} g(\sigma) \stackrel{(11)}{=} \bigcap_{\sigma \in S} h(\sigma) \stackrel{(6)}{=} h\left(\bigcap S\right)$$

The \bigcup and subtraction cases are analogous, via (5) and (9). \square

LEMMA 4 *Distinct paths are disjoint: if $\gamma, \gamma' \in P$ then*

$$\gamma \neq \gamma' \implies \gamma \cap \gamma' = \emptyset \quad (12)$$

Proof. $\gamma \cap \gamma'$ is a synchronisation by intersection-closure (Lemma 3). If $\gamma \neq \gamma'$ and $\gamma \cap \gamma' \neq \emptyset$ then $\gamma \cap \gamma'$ is a synchronisation strictly smaller than at least one of γ or γ' , contradicting minimality. \square

LEMMA 5 (DECOMPOSITION) *Every synchronisation σ is the disjoint union of its paths:*

$$\sigma = \biguplus \{ \gamma \subseteq \sigma : \gamma \text{ is a path} \} \quad (13)$$

Proof. Paths are disjoint by the previous lemma, so it remains to show that every link $c \in \sigma$ is in some (necessarily unique) path γ_c . (Automatically $\gamma_c \subseteq \sigma$, by minimality with respect to $\gamma_c \cap \sigma$.) Define

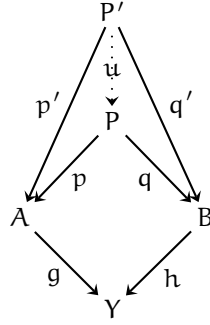
$$\gamma_c = \bigcap \{ \tau : \tau \text{ is a synchronisation and } c \in \tau \}, \quad (14)$$

a synchronisation by intersection-closure (Lemma 3) and non-empty since it contains σ . We must show that γ_c is minimal among all non-empty synchronisations (not merely among those containing c). Suppose $\mu \subsetneq \tau$ is a non-empty synchronisation. Let $\bar{\mu} = \tau \setminus \mu$, a synchronisation by subtraction-closure (Lemma 3). Then one of μ and $\bar{\mu}$ is a synchronisation containing c which is strictly smaller than γ_c , a contradiction. \square

Proof of Theorem 1. The square (1) commutes:

$$g(p\langle \alpha, \beta \rangle) \stackrel{(3)}{=} g(\alpha) \stackrel{(2)}{=} h(\beta) \stackrel{(4)}{=} h(q\langle \alpha, \beta \rangle).$$

Suppose $A \xleftarrow{p'} P' \xrightarrow{q'} B$ yields an analogous commuting square: $gp' = hq'$.



For $d \in P'$ let

$$\sigma(d) = p'(d) + q'(d) \subseteq A + B \quad (15)$$

which is a synchronisation since $gp' = hq'$. Define $u : P' \rightarrow P$ by taking $u(d)$ as the set of all paths within $\sigma(d)$:

$$u(d) = \{\gamma \in P : \gamma \subseteq \sigma(d)\} \quad (16)$$

Claim: u is injective. If $u(d) \cap u(e) \neq \emptyset$ there exists a path γ such that $\gamma \subseteq \sigma(d) \cap \sigma(e)$, say $\gamma = \alpha + \beta$ with $\alpha \subseteq A$ and $\beta \subseteq B$. Hence $\alpha \subseteq p'(d) \cap p'(e)$ and $\beta \subseteq q'(d) \cap q'(e)$. Since γ is a path, it is non-empty, so α or β is non-empty, say α . Thus $d = e$ by injectivity of p' . ■

Claim: $pu = p'$ and $qu = q'$. Suppose $a \in p'(d)$. Let γ be the unique path such that $a \in \gamma$ and $\gamma \subseteq \sigma(d)$, existing by Lemma 5. Then $\gamma \in u(d)$ (by (16)) and $a \in p(\gamma)$ (since $a \in A$ and p projects subsets of $A + B$ to subsets of A), hence $a \in p(u(d))$, so $p' \subseteq pu$.

Conversely, suppose $a \in p(u(d))$, i.e., there exists $\gamma \in P$ such that $a \in p(\gamma)$ and $\gamma \in u(d)$. By (16) we have $\gamma \subseteq \sigma(d)$, so $a \in p(\sigma(d))$, by (8). Since $p(\sigma(d)) = p'(d)$ (because p projects) we have $a \in p'(d)$. Hence $pu \subseteq p'$.

Since $p' \subseteq pu$ and $pu \subseteq p'$, we have $p' = pu$, whence $q' = qu$, by symmetry. ■

Finally, we must prove that u is unique, i.e., the commuting triangles $pu = p'$ and $qu = q'$ determine u . Let $\hat{u} : P' \rightarrow P$. Given $r : A \rightarrow M$ and $s : A \rightarrow N$ write $[r, s]$ for the corresponding injective relation $A \rightarrow M + N$. Thus $p' = p\hat{u}$ and $q' = q\hat{u}$ iff $[p, q]\hat{u} = [p', q']\hat{u}$. Paths are non-empty, so $[p, q]$ is total, hence monic (Lemma 1). Therefore $[p, q]\hat{u} = [p, q]u$ implies $\hat{u} = u$. □

References

- [Abr05] Abramsky, S. *Abstract scalars, loops, and free traced and strongly compact closed categories*. In *Proc. CALCO'05*, volume 3629 of *Lec. Notes in Comp. Sci.*, pp. 1–31. Springer, 2005.
- [Abr07] ———. *Temperley-lieb algebra: from knot theory to logic and computation via quantum mechanics*. In *Proc. Mathematics of Quantum Computing and Technology '05*, pp. 515–558. Tayler and Francis, 2007.
- [Bén67] Bénabou, J. *Introduction to bicategories*. In *Reports of the Midwest Category Seminar*, volume 47 of *Lecture Notes in Mathematics*, pp. 1–77. Springer-Verlag, 1967.
- [Bra37] Brauer, R. *On algebras which are connected with the semisimple continuous groups*. *Annals of Math.*, 38:857–872, 1937.
- [Gir87] Girard, J.-Y. *Linear logic*. *Theoretical Computer Science*, 50:1–102, 1987.

- [Gir89] ———. *Towards a geometry of interaction*. In *Categories in Computer Science and Logic*, volume 92 of *Contemporary Mathematics*, pp. 69–108, 1989. Proc. of June '87 meeting in Boulder, Colorado.
- [HG03] Hughes, D. J. D. & R. J. v. Glabbeek. *Proof nets for unit-free multiplicative additive linear logic (Extended abstract)*. In *Proc. LICS'03*, pp. 1–10. IEEE, 2003.
- [HG05] ———. *Proof nets for unit-free multiplicative-additive linear logic*. *ACM Transactions on Computational Logic (TOCL)*, 6:784–842, October 2005. Invited submission Nov. 2003, revised Jan. 2005, full version of [HG03].
- [HR05] Halverson, T. & A. Ram. *Partition algebras*. *European J. Combinatorics*, 26:869–921, 2005.
- [Jon94] Jones, V. F. R. *The potts model and the symmetric group*. In *Proc. Taniguchi Symposium on Operator Algebras (Kyuzeso '93)*, pp. 259–267, River Edge, NJ, 1994. World Sci. Pub.
- [Kau04] Kauffman, L. H. *Knot diagrammatics*. *arXiv:math/0410329v5*, 2004.
- [KL80] Kelly, G. M. & M. L. Laplaza. *Coherence for compact closed categories*. *J. Pure Appl. Algebra*, 19:193–213, 1980.
- [Mac71] Mac Lane, S. *Categories for the Working Mathematician*. Springer-Verlag, 1971.
- [Mar94] Martin, P. *Temperley-Lieb algebras for nonplanar statistical mechanics — the partition algebra construction*. *J. Knot Theory Ramifications*, 3:51–82, 1994.
- [TL71] Temperley, N. & E. Lieb. *Relations between the percolation and colouring problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the percolation problem*. In *Proc. Royal Society Series A*, volume 322, pp. 251–280, 1971.